Convex curves and a Poisson imitation of lattices

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Abstract

We solve a randomized version of the following open question: is there a strictly convex, bounded curve $\gamma \subset \mathbb{R}^2$ such that the number of rational points on γ, with denominator n, approaches infinity with n? Although this natural problem appears to be out of reach using current methods, we consider a probabilistic analogue using a spatial Poisson-process that simulates the refined rational lattice $\frac{1}{d}\mathbb{Z}^2$, which we call M_d, for each natural number d. The main result here is that with probability 1 there exists a strictly convex, bounded curve γ such that $|\gamma \cap M_d| \to +\infty$, as d tends to infinity. The methods include the notion of a generalized affine length of a convex curve, defined in [2].

1 Introduction

We first recall a natural and as yet unsolved problem from [2], that arose in the context of the geometry of numbers:

Problem 1. *Does there exist a strictly convex bounded curve γ in the plane such that, as n tends to infinity,*

$$\left|\gamma \cap \frac{1}{n}\mathbb{Z}^2\right| \to \infty? \tag{1}$$

The problem we address here is a probabilistic analogue of problem (1), for which we first give some motivation and definitions. We recall from [2] the following results, giving lower bounds for rational points on any strictly convex curve γ in the plane:

$$\left|\gamma \cap \bigcup_{k=1}^{n} \frac{1}{k}\mathbb{Z}^2\right| = O(n), \tag{2}$$

$$\sum_{k=1}^{n}\left|\gamma \cap \frac{1}{k}\mathbb{Z}^2\right| = O(n\log n). \tag{3}$$

The latter lower bound means that in the case of a positive answer to problem 1, the number of points of $\frac{1}{n}\mathbb{Z}^2$ on γ cannot converge to infinity faster than $\log n$. We also recall that in higher dimensions there is no gap in such asymptotic estimates. Namely, it is proved in [2] that for a bounded, closed, and strictly convex surface Γ in $\mathbb{R}^d$, with $d \geq 3$, we have

$$\liminf \frac{|\Gamma \cap n^{-1}\mathbb{Z}^d|}{n^{d-2}} < \infty.$$

As added motivation, we note that there are probabilistic models of rigid arithmetical objects, like the integers or the primes, which capture many quantitative characteristics of the original rigid structure. These probabilistic analogues are often easier to handle and offer support for the validity of their corresponding “rigid” statements. For instance, the Riemann Hypothesis may be considered as a statement that the Möbius μ-function displays some quasi-random behavior.

Specifically, the following asymptotic for the partial sums of the Möbius function $\mu(n)$ is known to be equivalent to RH: $\sum_{n\leq x}\mu(n) = o(x^{1/2+\varepsilon})$ for any $\varepsilon > 0$. However, if we replace the deterministic sequence $\{\mu(1), \mu(2), \dots\}$ by independent identically distributed bounded variables with zero mean, then such an estimate holds with probability 1, by standard methods. Because there is no evidence that the values of the Möbius function are highly dependent, this asymptotic is considered as heuristic support for the Riemann Hypothesis.

Here we deal with another arithmetical object, the integer lattice $\mathbb{Z}^d$ in d-dimensional Euclidean space. We consider its natural probabilistic analogue: a random configuration of points given by a Poisson process with intensity 1 in $\mathbb{R}^d$.

As a non-example for problem (1), consider the unit circle S, centered at the origin. Then $\frac{1}{n}\sum_{k=1}^{n}|S\cap n^{-1}\mathbb{Z}^2|$ tends to infinity. In other words, on average the unit circle has many rational points. However, for any prime n the unit circle has only at most 12 rational points with denominator n, and hence the circle does not satisfy property (1). Moreover, if we consider any real algebraic curve γ in the plane, of genus $g \geq 2$, we know by Faltings' theorem that it contains only finitely many rational points. Hence all such curves γ do not satisfy (1). Though we do not know of any formal argument that allows us to resolve problem 1, these counterexamples suggest a negative answer to this problem.

On the other hand, "most" convex curves are not algebraic, and if we pick a convex curve γ "at random", it is natural to ask if γ might still satisfy property (1)?

In sharp contrast with algebraic curves, it turns out that if we work in the probabilistic framework suggested above, then we find here an affirmative answer to a very natural probabilistic analogue of problem 1.

To describe the main result, we first order all of the prime powers in ascending order, as follows: $q_1 = 2, q_2 = 3, q_3 = 2^2, q_4 = 5, q_5 = 7, q_6 = 2^3, q_7 = 3^2, q_8 = 11, q_9 = 13, q_{10} = 2^4$, and so on. For technical reasons that are explicated in the preliminaries section 2 below, we define $w_{q_k} := q_k^2\left(1-\frac{1}{p^2}\right)$, where q_k is a power of the prime p. The value w_{q_k} will be the intensity of our Poisson process.

We define our randomized analogue of a lattice to be a Poisson point set M_{q_k}, with intensity equal to w_{q_k}, for each q_k which is a power of a prime. In other words, the randomized point set M_{q_k} by definition satisfies a spatial Poisson process distribution. It will turn out that the only property we really require in our analysis is that the intensity $w_{q_k} > \frac{q_k^2}{2}$. In section 2, we define a more general Poisson analogue of a lattice, called M_d for any integer d, but we show that due to the elementary combinatorial structure of M_d, it is sufficient to work with the particular analogue M_{q_k} defined here.

Theorem 1. *With probability* 1 *there exists a strictly convex, bounded curve* γ *such that*

$$|\gamma\cap M_{q_k}| \to +\infty,$$

as k *tends to* ∞.

Many geometric questions regarding lattices appear to have similar answers to their corresponding Poisson framework questions, in the context of a 'perturbed lattice'. Here are some more examples.

Let K be a convex and compact set in the plane. We consider all possible convex polygons with vertices in $\frac{1}{n}\mathbb{Z}^2\cap K$ for some large n. We may now ask some natural and intuitive questions:

Problem 2. *how many such polygons are there in total?*

Problem 3. *How many vertices does such a polygon have?*

Problem 4. *What does a typical polygon look like?*

The same questions may be asked for a random set of points instead of points chosen from a lattice. For example, we may consider $C(K)\cdot n^2$ independetly and uniformly distributed points

in K, for some constant $C(K)$, independent of n. We may also consider a Poisson process inside K, with intensity n^2.

The answers to above questions (2), (3), and (4) appear to be very similar in the lattice setting and in the probabilistic setting (see [3], [4], [6]). Amazingly, only the specific values of constants differ. However, the methods of the geometry and numbers (lattice setting) and of stochastic geometry (randomization) do indeed differ. In this paper we work with a randomized model for the set of rational points $\mathbb{Q}^2$.

2 Preliminaries

First, we use the Poisson process of intensity 1 in $\mathbb{R}^2$, which we call $\mathcal{Z}$, as our most natural realization for a randomized relaxation of the integer lattice $\mathbb{Z}^2$. In general, we denote by $\mathcal{P}^2(n)$ a Poisson process of intensity n. We recall the definition of this spacial process. For more information, see, e.g. [8]

Definition 1. A Poisson process of intensity λ is characterized by the following two properties:

- For any region A with area $|A|$, the number of events in A obeys a 1-dimensional Poisson distribution with mean $\lambda|A|$.
- The number of events in any finite collection of non-overlapping regions are independent of each other.

In order to extend Poisson process to the rational set $\mathbb{Q}^2$ one might first represent the original lattice by the union $\cup_{n=1}^{\infty}\frac{1}{n}\mathbb{Z}^2$ and then define the randomized rational lattice as $\cup_{n=1}^{\infty}\frac{1}{n}\mathcal{Z}^2$. Unfortunately, in this case the probabilistic object does not enjoy the set-theoretical property of the original lattice, namely that $\frac{1}{n}\mathbb{Z}^2 \subset \frac{1}{m}\mathbb{Z}^2$ for $n|m$. For this reason we introduce a slightly different randomization process. We begin with a disjoint decomposition of $\mathbb{Q}^2$ given as follows.

$$\begin{gathered} \mathbb{Q}^2 = \bigcup_{n=1}^{\infty} L'_n, \text{ where} \\ L'_n := \{(a/n, b/n), gcd(a,b,n) = 1\}. \end{gathered} \tag{4}$$

We notice that each set L'_n can be expressed by inclusion-exclusion principle as a sum (with signs) of scaled integer lattices. In particular, if $p_1, \ldots, p_k$ are all prime divisors of n, then

$$L'_n = \frac{1}{n}\mathbb{Z}^2 - \frac{1}{n/p_1}\mathbb{Z}^2 - \cdots - \frac{1}{n/p_k}\mathbb{Z}^2 + \frac{1}{n/p_1p_2}\mathbb{Z}^2 + \frac{1}{n/p_1p_3}\mathbb{Z}^2 + \cdots + \frac{(-1)^k}{n/p_1p_2\cdots p_k}\mathbb{Z}^2.$$

Thus in the randomization process it is natural to assume that points in each L'_n are distributed uniformly in the plane with the following density per unit area

$$w_n := n^2 \prod_{\substack{p|n,\\ p\, is\, prime}} (1 - 1/p^2).$$

We notice that $\frac{1}{n}\mathbb{Z}^2 = \bigcup\limits_{d|n} L'_d$. We define M_n as a random Poisson configuration of intensity w_n. Then the random analogue of $\mathbb{Q}^2$ is

$$\mathcal{Q}^2 := \bigcup_{n=1}^{\infty} M_n$$

(processes with different n are mutually independent). We observe a few useful properties of such a process.

- $\bigcup\limits_{d|n} M_d$ coincide with the usual Poisson process of intensity n^2, which is the standard randomized analog for the rational lattice $\frac{1}{n}\mathbb{Z}^2$ with fixed denominator n.

- $\bigcup\limits_{d|n_1} M_d \cap \bigcup\limits_{d|n_2} M_d = \bigcup\limits_{d|\gcd(n_1,n_2)} M_d$ and similarly for lattices $\frac{1}{n_1}\mathbb{Z}^2 \cap \frac{1}{n_2}\mathbb{Z}^2 = \frac{1}{\gcd(n_1,n_2)}\mathbb{Z}^2$.
  In this sense, the family of $\bigcup\limits_{d|n} M_d$ obeys the same set theoretical structure as the rational lattices.
- $\bigcup\limits_{k=1}^{n} M_k$ is also a Poisson process of certain intensity; its expected number of points coincides with the number of rational points with denominators not exceeding n in any unit square of general position, i.e. $\bigcup\limits_{k=1}^{n} M_k$ is a natural analog of $\bigcup\limits_{k=1}^{n} \frac{1}{k}\mathbb{Z}^2$.

3 Results about generalized affine length

The proof uses the notion of a *generalized affine length of a convex chain* introduced in [1]. Let $S(F)$ denote the doubled area of a polygon F, $\mathbf{x} \times \mathbf{y}$ denote the pseudo-scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$ (i.e. an oriented area of a parallelogram, based on these vectors).

Fix a triangle ABC oriented so that $S = S(ABC) = +\overline{AC} \times \overline{CB}$.

Let $\gamma = AC_1C_2 \dots C_k B$ be a strictly convex chain. We call it (AB, C)-chain, if all its vertices lie inside triangle ABC.

Let (AB, C)-chain $\gamma = AC_1C_2 \dots C_k B$ be inscribed in an (AB, C)-chain $\gamma_1 = AD_1D_2 \dots D_{k+1}B$ (i.e. points C_i lie on respective segments D_iD_{i+1} ($i = 1, 2, \dots, k$)). Define a *generalized affine length* of a chain γ with respect to γ_1 as

$$l_A(\gamma : \gamma_1) := \sum_{i=0}^{k} S(C_iD_{i+1}C_{i+1})^{1/3} \qquad (C_0 = A,\ C_{k+1} = B), \tag{5}$$

Part (i) of the following lemma is well-known, and part (ii) is a non-surprising quantitative improvement of (i).

Lemma 1. *(i) Let points P, R be chosen on the sides AC and BC of the triangle ABC respectively, and a point Q — on the segment PR. Then*

$$S(AQP)^{1/3} + S(BQR)^{1/3} \le S^{1/3}.$$

(ii) For fixed $0 < \alpha < S^{1/3}$ call a point Q inside $\triangle ABC$ α-admissible if there exist points P, R on AC, BC respectively such that Q lies on a segment PR and

$$S^{1/3} - S(AQP)^{1/3} - S(BQR)^{1/3} \le \alpha.$$

Then the area of the set of α-admissible points is not less than $\frac{1}{8}\sqrt{\alpha}S^{5/6}$.

Proof. We denote

$$\varepsilon = \frac{\alpha}{S^{1/3}},\ 0 < \varepsilon < 1,$$

and

$$\mathrm{Err} = 1 - \left(\frac{\mathrm{S(APQ)}}{\mathrm{S}}\right)^{1/3} - \left(\frac{\mathrm{S(BQR)}}{\mathrm{S}}\right)^{1/3}.$$

Our goal is to show that

(i) $\mathrm{Err} \ge 0$, and

(ii) Set of points Q, for which $\mathrm{Err} \le \varepsilon$, has area at least $\frac{1}{8}\sqrt{\alpha}S^{5/6} = \frac{1}{8}\sqrt{\varepsilon}S$.

By simple geometrical observations,

$$\begin{aligned}
\mathrm{Err} &= 1 - \left(\frac{S(APQ)}{S}\right)^{1/3} - \left(\frac{S(BQR)}{S}\right)^{1/3} \\
&= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \left(\frac{AP}{AC}\cdot\frac{PQ}{PR}\cdot\frac{RC}{BC}\right)^{1/3} - \left(\frac{PC}{AC}\cdot\frac{QR}{PR}\cdot\frac{BR}{BC}\right)^{1/3} \\
&= \left(\frac{1}{3}\left(\frac{AP}{AC} + \frac{PQ}{PR} + \frac{RC}{BC}\right) - \left(\frac{AP}{AC}\cdot\frac{PQ}{PR}\cdot\frac{RC}{BC}\right)^{1/3}\right) + \\
&\quad + \left(\frac{1}{3}\left(\frac{PC}{AC} + \frac{QR}{PR} + \frac{BR}{BC}\right) - \left(\frac{PC}{AC}\cdot\frac{QR}{PR}\cdot\frac{BR}{BC}\right)^{1/3}\right) \qquad (6)
\end{aligned}$$

Later is the sum of two expressions of the form $(x+y+z)/3 - (xyz)^{1/3}$. Hence the proof of (i) is finished by applying AM-GM inequality.

For the proof of (ii) we want first to estimate both of the aforementioned expressions in (6) from above. We will do it in a particular case when all ratios involved in (6) lie between $1/8$ and 1 and differ pairwise by at most 2δ, where δ is a parameter, value for which we will assign later.

We note that for any non-negative numbers x, y, z the following identity holds:

$$\begin{aligned}
&(x+y+z)/3 - (xyz)^{1/3} = \\
=&\frac{1}{6}(x^{1/3} + y^{1/3} + z^{1/3})((x^{1/3} - y^{1/3})^2 + (y^{1/3} - z^{1/3})^2 + (z^{1/3} - x^{1/3})^2). \qquad (7)
\end{aligned}$$

Also, when x, y, z lie between $1/8$ and 1 and differ pairwise by at most 2δ,

$$\left|x^{1/3} - y^{1/3}\right| = \frac{|x-y|}{\left|x^{2/3} + y^{2/3} + x^{1/3}y^{1/3}\right|} \le \frac{2\delta}{3(\frac{1}{8})^{2/3}} = \frac{8\delta}{3},$$

and we can use this to estimate right hand side of (7) from above:

$$\begin{aligned}
&\frac{1}{6}(x^{1/3} + y^{1/3} + z^{1/3})((x^{1/3} - y^{1/3})^2 + (y^{1/3} - z^{1/3})^2 + (z^{1/3} - x^{1/3})^2) \le \\
\le&\frac{1}{6}\cdot 3 \cdot 3(\frac{8\delta}{3})^2 = \frac{32}{3}\delta^2
\end{aligned}$$

Now, fix $\delta = \sqrt{\varepsilon}/8$. It immediately follows that

$$Err \le 2\frac{32}{3}\delta^2 \le 2\cdot\frac{32}{3}\cdot(\frac{\sqrt{\varepsilon}}{8})^2 \le \varepsilon,$$

in other words, if all ratios in (6) lie between $1/8$ and 1 and differ pairwise by at most $2\delta = \sqrt{\varepsilon}/4$, then Q is ε-admissible. It remains to prove that the locus of such points Q has area at least $\delta S = \frac{1}{8}\sqrt{\varepsilon}S$.

The statement of (ii) is preserved under affine transforms, so without loss of generality, suppose $A = (-1, 1)$, $B = (1, 1)$, $C = (0, -1)$, and therefore $S = 2$. We will now produce a set of points Q of area $\delta S = 2\delta$, for which desired conditions on ratios of segments are satisfied.

Consider the point $Q = (t, t^2 + \tau)$, $-1/2 \le t \le 1/2$, $-\delta \le \tau \le \delta$. Draw a line $y = 2tx - t^2 + \tau$ through Q and let it meet sides AC, BC in points P, R respectively. By choice of $\delta = \sqrt{\varepsilon}/8 \le 1/8$, all such points Q indeed lie inside triangle ABC. The locus of all such points Q has area exactly 2δ. Also with this choice of t and τ, points P and R lie on sides AC and BC respectively, not on the side extensions, and so we indeed get a legitimate triple P, Q, R. It remains to show that with this choice of points P, Q, R, desired conditions on ratios of segments are satisfied, which will finish the proof.

Direct calculations show that the ratios $BR : BC$, $QR : PR$, $PC : AC$ are close to $(1+t)/2$ with accuracy δ, and therefore differ pairwise by at most 2δ. Also, since $\delta = \sqrt{\varepsilon}/8 \le 1/8$ and

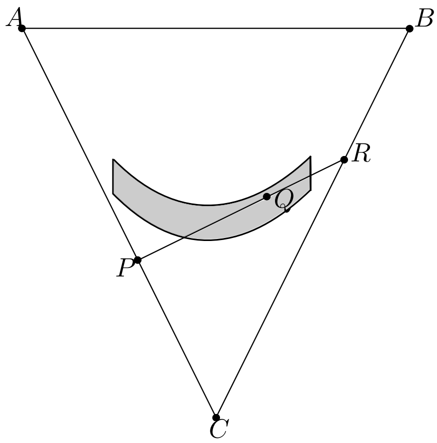

Figure 1: The shaded area consists of some ε-admissible points inside a triangle ABC.

$-1/2 \le t \le 1/2$, it follows that all these ratios are in range from $(1-\frac{1}{2})/2 - \frac{1}{8}$ to $(1+\frac{1}{2})/2 + \frac{1}{8}$, and therefore lie between $1/8$ and 1. Similarly, the ratios $RC : BC$, $PQ : PR$, $AP : AC$ are close to $(1-t)/2$ with accuracy δ, so they also differ pairwise by at most 2δ and lie between $1/8$ and 1, which finishes the proof. $\square$

Remark. Constant $\frac{1}{8}$ in the second part of Lemma 1 is not tight. For the purpose of this work we only need the area to be $\gtrsim \sqrt{\alpha} S^{5/6}$ and do not elaborate on the tight value for the constant, however this might be of independent interest.

4 A construction of some convex polygonal curves

Throughout the paper, we adopt the following notation. We write $f(n) \gtrsim g(n)$ to mean that $f(n)$ grows asymptotically at least as fast as the function $g(n)$, as n tends to infinity. Formally, if $f(n)$ and $g(n)$ are two non-negative functions on integers, $f(n) \gtrsim g(n)$ if and only if there exist some positive constants C and N, such that for each $n > N$, we have $f(n) \ge Cg(n)$.

In this section we first give an overview of the ideas that are used in the proof of the main theorem, and we begin by constructing some polygonal curves whose limit will later be our strictly convex bounded curve γ. We define inductively a sequence of convex (AB, C)-chains γ_n. We start with the chain $\gamma_1 = AB$. Then at each step we modify γ_n to γ_{n+1} by appending one new point Q to it, where Q comes from our pseudo-lattice, in such a way that γ_{n+1} is a convex (AB, C)-chain and the probability that Q is in M_{q_n} is sufficiently high.

We also construct an auxiliary (AB, C)-chain γ'_n, such that at each step γ_n is inscribed in γ'_n. We will keep track of two quantities. The first quantity is ℓ_n, the generalized affine perimeter of γ_n with respect to γ'_n, defined by (5). It will be useful to prove that under some conditions ℓ_n is decreasing, but is bounded by positive constant from below. The second quantity is s_n, the area of locus of points Q lying between γ_n and γ'_n, such that these points can be added to γ_n on the n'st step while not causing ℓ_n to decrease too rapidly. Finally, we will estimate s_n from below, as well as show that s_n is decreasing. This lower bound on s_n will give us a sufficiently high probability for choosing a new point Q from M_{q_n} at the n'th step of our construction, for each $n \ge N$.

We now begin the construction of γ_{n+1} from γ_n, while emphasizing the importance of choosing Q appropriately. We start with a pair of $(AB; C)$-chains $\gamma_1 = AB$ and $\gamma'_1 = ACB$. At step n we have a pair of (AB, C)-chains γ_n and γ'_n, γ_n being inscribed in γ'_n,

$$\gamma_n = AC_1C_2 \dots C_{n-1}B,$$
$$\gamma'_n = AD_1D_2 \dots D_nB,$$

where C_i lies on D_iD_{i+1} for each i. We also denote triangles $\Delta_i = C_{i-1}D_iC_i$, $i = 1, 2, \dots, n$, where $C_0 = A$, $C_n = B$, and their (doubled) areas as $S_i = S(\Delta_i)$. Finally, we denote the affine

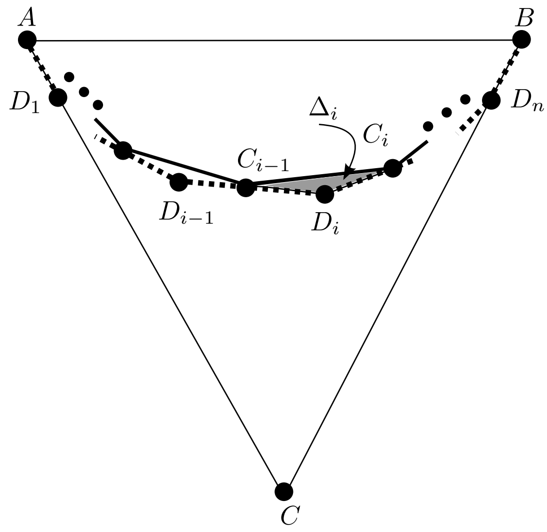

Figure 2: A triangle ABC with two convex (AB, C)-chains γ_n (bold) and γ'_n (dashed bold) inside. At the i'th step, a small triangle Δ_i is constructed.

length of γ_n w.r.t. γ'_n as

$$\ell_n = l_A(\gamma_n : \gamma'_n) = \sum_{i=1}^{n} S_i^{1/3}. \tag{8}$$

Suppose the point Q is chosen inside a triangle Δ_i and points P and R are chosen respectively on the line segments $C_{i-1}D_i$, D_iC_i, such that P, Q, R are collinear. Then we define a new pair of convex (AB, C)-chains γ_{n+1} and γ'_{n+1} as follows:

$$\begin{aligned} \gamma_{n+1} &= AC_1 \dots C_{i-1}QC_i \dots C_{n-1}B, \\ \gamma'_{n+1} &= AD_1 \dots D_{i-1}PRD_{i+1} \dots D_nB. \end{aligned}$$

A point $Q \in \bigcup_{i=1}^{n} \Delta_i$ is called ***α-admissible*** if $\ell_n - \ell_{n+1} \le \alpha$, where ℓ_{n+1} is now defined according to (8) using γ_{n+1} and γ'_{n+1}. We note that this definition is consistent with the definition given in the second part of Lemma 1.

The following Lemma shows that ℓ_n is bounded from below.

Lemma 2. *Suppose a sequence ℓ_n has the recursion property that $\ell_n - \ell_{n+1} \le a_n$, where the sequence a_n is defined by*

$$a_n = \begin{cases} \frac{1}{2}\ell_n & \text{if } n = 0, 1, \text{or } 2 \\ 2\ell_n n^{-1}(\log n)^{-3/2} & \text{if } n \ge 3. \end{cases}$$

Then $\ell_n \gtrsim \ell_0$.

Proof. Indeed, for each $0 \le k \le n-1$ we have $\ell_k - \ell_{k+1} \le a_k$, and with formula for a_k, we can rewrite this inequality as

$$\ell_k - \ell_{k+1} \le 2\ell_k k^{-1}(\log k)^{-3/2},$$

which can be again rewritten as

$$\ell_{k+1} \ge \ell_k \left(1 - \frac{2}{k \log^{3/2} k}\right).$$

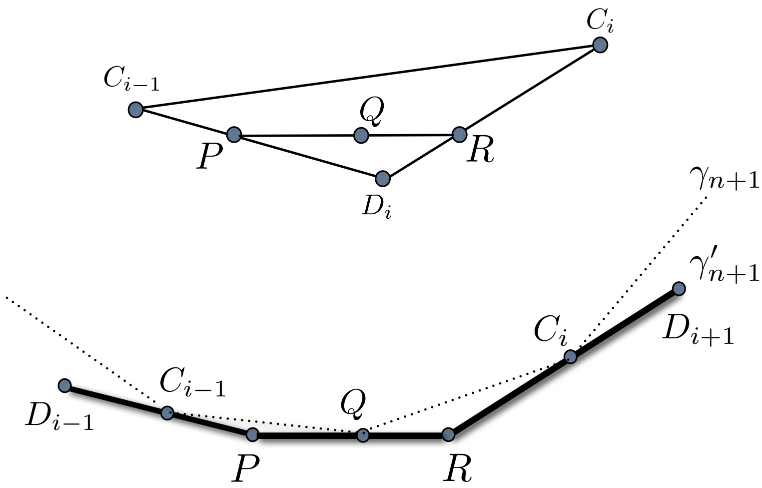

Figure 3: At the n'th step of our inductive construction, we choose an a_n-admissible point Q (above) inside a little triangle Δ_i, and then we use the Poisson-point Q to construct two new convex chains γ_{n+1} and γ'_{n+1}.

Multiplying such inequalities for each $0 \le k \le n-1$, we get

$$\ell_n \ge \ell_0 \prod_{k=0}^{n-1} \left(1 - \frac{2}{k \log^{3/2} k}\right).$$

Let us denote the product on a right hand side as X. In order to show $\ell_n \gtrsim \ell_0$, it is now sufficient to show that X is bounded by a positive constant from below. Since $0 \le X \le 1$, this is equivalent to showing that $\log X$ is bounded by some constant from below. We can expand $\log X$:

$$\log X = \log \prod_{k=0}^{n-1} \left(1 - \frac{2}{k \log^{3/2} k}\right) = \sum_{k=0}^{n-1} \log \left(1 - \frac{2}{k \log^{3/2} k}\right) \ge \sum_{k=0}^{n-1} \left(-\frac{1}{k \log^{3/2} k}\right).$$

The last inequality is true due to the fact that $\log(1-2x)$ is greater than $-x$ for all $0 \le x \le 0.5$, and the fact that for each $k > 0$ all of the real numbers $\frac{1}{k \log^{3/2} k}$ lie in $[0, 0.5]$. Thus $\log X$ is bounded from below, because the corresponding series $\sum_{k=0}^{n-1} \left(\frac{1}{k \log^{3/2} k}\right)$ converges, and the proof of the Lemma is complete.

$\square$

If for each $1 \le i \le n$, the i'th step in our process has a corresponding point Q which is a_i-admissible, where a_i is defined as in Lemma 2, then by the conclusion of Lemma 2 we have

$$\ell_n \gtrsim \ell_0. \tag{9}$$

Now we specify the choice of a point Q at the n'th step of our inductive process. We choose Q from $\bigcup_{i=1}^{n} \Delta_i$ by using the following recipe. If there exists a point of a Poisson process M_{q_n} that is a_n-admissible, then we let Q be any such point. Otherwise, we let Q be any a_n-admissible point.

We define s_n to be the measure (area) of the set of all points $Q \in \bigcup_{i=1}^n \Delta_i$, such that Q is a_n-admissible. Next, we estimate s_n from below.

Lemma 3.

$$s_n \gtrsim \ell_0^3 n^{-2} \log^{-3/4} n. \tag{10}$$

Proof. To prove this, we start from partitioning the set of indices $\{1, 2, \dots, n\}$ into two sets G and B (good and bad), where $i \in B$ if $S_i^{1/3} \le a_n$ and $i \in G$ if $S_i^{1/3} > a_n$. Note that $\sum_{i \in B} S_i^{1/3} \le n a_n \le \ell_n/2$ by choice of a_n, so

$$\sum_{i \in G} S_i^{1/3} \ge \ell_n/2. \tag{11}$$

For any good index i (assume $|G| = k$, so there are k good indices) we can apply Lemma1,(ii), to triangle Δ_i: the set of a_n-admissible points in Δ_i has area at least

$$\frac{1}{8}\sqrt{a_n} S_i^{5/6}$$

Then the total area s_n of the set of a_n-admissible points can be estimated as follows:

$$\begin{aligned}
s_n \ge \sum_{i \in G} \frac{1}{8}\sqrt{a_n} S_i^{5/6} = \frac{1}{8}\sqrt{a_n} k \frac{\sum_{i \in G} S_i^{5/6}}{k} &\gtrsim \\
\sqrt{a_n} k \left(\frac{\sum_{i \in G} S_i^{1/3}}{k} \right)^{\frac{5}{6} : \frac{1}{3}} &\ge \\
\sqrt{a_n} k \left(\frac{\ell_n}{2k} \right)^{5/2} &\gtrsim \\
\sqrt{a_n} \ell_n^{5/2} k^{-3/2} &\ge \\
\sqrt{a_n} \ell_n^{5/2} n^{-3/2} &= \\
\left(C_0 \ell_n n^{-1} \log^{-3/2} n \right)^{1/2} \cdot \ell_n^{5/2} n^{-3/2} &\gtrsim \\
\ell_n^3 n^{-2} \log^{-3/4} n &\gtrsim \\
\ell_0^3 n^{-2} \log^{-3/4} n&.
\end{aligned}$$

(our first inequality follows from power mean estimate of $\sum S_i^{5/6}$ via $\sum S_i^{1/3}$, second inequality follows from (11), and the last inequality follows from (9)). □

Remark. s_n is bounded from above by the area of ABC.

Lemma 4. *Probability that a given triangle Δ is never divided in our process is* 0.

Proof. Assume that triangle Δ is never divided after step N, we want to find the probability of such event. We note that area A of Δ remains constant throughout all the steps $n > N$. Probability that on step n the point Q is chosen from a triangle Δ_i equals $\frac{A_n}{s_n}$, where A_n is the contribution of Δ to s_n. So the probability that Δ is never divided after N steps equals

$$\prod_{n > N} \left(1 - \frac{A_n}{s_n} \right). \tag{12}$$

Showing that this product is 0 is equivalent to showing that its inverse is infinity. But the inverse expression can be estimated from below as follows:

$$\prod_{n>N}\left(\frac{1}{1-\frac{A_n}{s_n}}\right) \geq \prod_{n>N}\left(1+\frac{A_n}{s_n}\right) \geq \sum_{n>N}\frac{A_n}{s_n} \geq \tag{13}$$

$$\sum_{n>N}\frac{\frac{1}{8}\sqrt{a_n}A^{5/6}}{S(ABC)} \geq \sum_{n>N} C\sqrt{n^{-1}\log^{-3/2} n}, \tag{14}$$

where C is some positive constant. The series in the last expression diverges, which ends the proof. □

The following geometrical lemma is a technical argument that is required to prove that length of segments of γ_n approaches 0 with probability 1.

Lemma 5. *Suppose we are in conditions of Lemma 1, and the additional property is satisfied: ratios $AP : AC$, $PQ : PR$, $BR : BC$ lie between $1/8$ and $7/8$. Let m be the maximum of lengths of the sides of ABC. Then lengths of all segments AP, PQ, QR, RB, AQ, QB are at most $\frac{399}{400}m$.*

Proof. From the bounds on ratios it is immediate that lengths of AP, PQ, QR, RB are at most $\frac{1}{8}m$. However estimating AQ and QB needs some more work. Let D be the point of intersection of AQ and BC. We apply Menelaus' theorem to triangle ACD and points P, Q, R:

$$\frac{AQ}{QD}\cdot\frac{DR}{RC}\cdot\frac{CP}{PA} = 1, \tag{15}$$

and to triangle PCR and points A, Q, D:

$$\frac{PA}{AC}\cdot\frac{CD}{DR}\cdot\frac{RQ}{QP} = 1. \tag{16}$$

From (16) we get

$$\frac{CD}{DR} = \frac{AC}{PA}\cdot\frac{QP}{RQ} \leq 8\cdot 7 = 56. \tag{17}$$

Now combining (15) with (17) we get

$$\frac{AQ}{QD} = \frac{RC}{DR}\cdot\frac{PA}{CP} = (\frac{CD}{DR}+1)\cdot\frac{PA}{CP} \leq (56+1)\cdot 7 = 399,$$

which finally gives us

$$AQ \leq 399QD \Rightarrow 400AQ \leq 399(AQ+QD) \Rightarrow AQ \leq \frac{399}{400}AD \Rightarrow AQ \leq \frac{399}{400}m.$$

Length of QB can be estimated similarly. □

Lemma 6. *Probability that γ contains a straight line segment is* 0.

Proof. We will first show that lengths of all segments $C_{i-1}D_i$, D_iC_i and $C_{i-1}C_i$ approach 0 with probability 1. We can additionally require that α-admissible points Q satisfy the following property: ratios $AP : AC$, $PQ : PR$, $BR : BC$ lie between 1/8 and 7/8. We can assume that, because the set of α-admissible points constructed in the proof of Lemma 1 satisfies this property. Our statement immediately follows from combining lemmas 4 and 5.

Now assume there is a straight line segment l in γ. Consider a small ball B with the center in the midpoint of this straight line segment. Lengths of all segments of γ_n approach 0, while γ_n themselves approach γ, so there should exist an infinite sequence of vertices Q_k of γ_{n_k} inside the ball B. By convexity of γ, all Q_k should lie on l, which contrdicts the fact that all γ_n are strictly convex. □

Lemma 7. *Fix any triangle ABC in the plane, and fix any $\varepsilon > 0$. Then with probability at least $1 - \varepsilon$, there exists a convex countable (AB, C)-chain γ with at least one point from each pseudo-lattice M_{q_n}, for some N with $n \geq N$.*

Proof. Consider now a Poisson random configuartion M_{q_n} on the plane. If q_n is a power of prime p, then intensity is $q_n^2 \cdot (1 - \frac{1}{p^2})$ points per unit area. Also we note that $q_n \gtrsim n \log n$, and $1 - \frac{1}{p^2} \geq \frac{1}{2}$. Denote the probability that there are no a_n-appropriate points w.r.t. γ_n by P_n. We can estimate P_n from above using (10):

$$
\begin{aligned}
P_n = \exp(-q_n^2(1 - \frac{1}{p^2})s_n) &\leq \exp\left(-n^2 \log^2 n \cdot C\ell_0^3 n^{-2} \log^{-3/4} n\right) \\
&= \exp\left(-C\ell_0^3 \log^{5/4} n\right) \\
&= n^{-C\ell_0^3 \log^{1/4} n}, \qquad (18)
\end{aligned}
$$

where C stands for a positive constant.

By (18) the series $\sum P_n$ does converge, which can be formally written as follows: for any $\varepsilon > 0$ there exists positive constant N, such that

$$\sum_{n \geq N} P_n < \varepsilon.$$

It means that for any $\varepsilon > 0$ and for any triangle ABC one may find a big constant N such that the probability to find a convex countable (AB, C)-chain with at least one point from each pseudo-lattice $L_{q_n}, n \geq N$ is not less than $1 - \varepsilon$. ☐

Proof of Theorem1. Fix $\varepsilon_0 > 0$. Consider the countable sequence of points $A_1 A_2 \dots$ on a circle, monotonically converging to some point A_0. Consider triangles, formed by chords $A_i A_{i+1}$ and tangents in its endpoints. Apply Lemma 7 for i-th triangle taking $\varepsilon_i = \varepsilon_0 / 2^i$. We get a series of convex countable chains γ^i, each of which has at least one point from each $L_{q_n}, n \geq N_i$ with probability at least $\varepsilon_0/2^i$. Note that the events in different triangles are independent, so the processes inside them do not depend on what have we found in other triangles. Finally we take γ as a union of γ^i over all i, note that γ is convex by the choice of corresponding triangles. By construction, $\gamma \cap L_{q_n} \to \infty$ with probability at least ε_0. ☐

5 Questions

There are many theorems and open questions concerning rational points on convex curves and surfaces, which may be settled for pseudo-lattices as well. For instance, consider the analogue of the main result of [1]: if γ is a bounded strictly convex curve in the Euclidean plane, then $|\gamma \cap \frac{1}{n}\mathbb{Z}^2| = o(n^{2/3})$.

The corresponding conjecture for pseudo-lattices is that with probability 1 for any sequence of independent Poisson spatial processes M_d with intensities $d = 1, 2, \dots$, the asymptotic bound $|\gamma \cap M_d| = o(d^{2/3})$ holds for any strictly convex curve γ.

At the moment, we do not even know whether such a probability measure exists, and the reason for the difficulty here is that γ is arbitrary, which makes it hard to prove measurability.

References

[1] F. V. Petrov. On the number of rational points on a strictly convex curve. Funk. Anal. Pril. **40** (1), 30-42 (2006). (Russian, English translation: Func. Anal. App. **40** (1), 24-33 (2006)).

[2] F. V. Petrov. Estimates for the number of rational points on convex curves and surfaces. Zap. Nauch. Sem. POMI **344**, 174-189 (2007). (Russian, English translation: J. of Math. Sci. **147** (6), 7218-7226 (2007)).

[3] A. M. Vershik. The limit form of convex integral polygons and related problems. Funkts. Anal. Prilozh. **28** (1), 16-25 (1994). (Russian, English translation: Funct. Anal. Appl. **28** (1), 13-20 (1994)).

[4] I. Barany, A. M. Vershik. On the number of convex lattice polytopes. Geom. Funct. Anal. **2** (4), 381-393 (1992).

[5] I. Barany, M. Reitzner. Poisson polytopes. Annals Prob. **38**, 1507-1531 (2010).

[6] I. Barany. Random points and lattice points in convex bodies, Bull. AMS, **45**, 339-365 (2008).

[7] I. Barany, V. H. Vu. Central limit theorems for Gaussian polytopes, Ann. Prob. **35**, 1593-1621 (2007).

[8] Daley, D.J, Vere-Jones, D. An Introduction to the Theory of Point Processes. Springer, New York. (1988)